TEACHERS' CONSTRUCTION OF DYNAMIC MATHEMATICAL MODELS BASED ON THE USE OF COMPUTATIONAL TOOLS

Aarón Reyes-Rodríguez
UAEH
aaronr@uaeh.edu.mx

Manuel Santos-Trigo
Cinvestav-IPN
msantos@cinvestav.mx

To what extent does the use of computational tools offer teachers the possibility of constructing dynamic models to identify and explore diverse mathematical relations? What ways of reasoning or thinking about the problems emerge during the models construction with the use of the tools? These research questions guided the development of the study that led us to document the process exhibited by high school teachers to model mathematical situations dynamically. In particular, there is evidence that the use of computational tools helped them identify and explore a set of mathematical relations dynamically. In this process, the participants had opportunity of fostering an inquisitive approach to models construction that values ways of formulating conjectures or mathematical relations and ways to support them.

Introduction

Models construction plays a fundamental role during the development of mathematical knowledge. In particular, the modeling cycle that involves examining the phenomenon to be modeled, identifying and discussing assumptions and elements to construct the model, and exploring and validating the model provides useful information to frame an instructional approach to foster teachers' practices and students' mathematical thinking. In this context, we argue that a central activity in students' process of developing mathematical concepts and solving problems is the construction of models that are used to identify, explore and support mathematical relations. Goldin (2008, p. 184) states that "…A model is a specific structure of some kind that embodies features of an object, a situation or a class of situations or phenomena – that which the model represents". How a model is constructed? How can one evaluate the pertinence of a model? What is the role of the use of computational tools in the construction of models? To respond to and discuss these questions, we identify an inquisitive or inquiry approach as a crucial activity associated with the modeling process.

> Mathematical modelling is the process of encountering an indeterminate situation, problematizing it, and bringing inquiry, reasoning, and mathematical structures to bear to transform the situation. The modelling produces an outcome –a model– which is a description or a representation of the situation, drawn from the mathematical disciplines, in relation to the person's experience, which itself had changed through the modelling process. (Confrey & Maloney, 2007, p. 60)

Teachers need to problematize their instructional practice in order to construct instructional routes. In this process, it is crucial that they get engaged into an inquisitive approach to examine the situation (formulation and discussion of questions) in terms of mathematical resources and strategies that lead them to the construction of models. A model then, is the vehicle for teachers to identify mathematical relations and to solve problems. We contend that the development and availability of computational tools offer teachers and students the possibility of enhancing their repertoire of heuristics strategies to deal with mathematical relations embedded in models. It is also important to recognize that different tools may offer distinct opportunities for them to represent and approach mathematical problems. For example, with the use of dynamic software, such as Cabri-Geometry or Sketchpad, some tasks can be modeled dynamically as a mean to identify and explore diverse mathematical relations or conjectures. Thus, tasks or problems are seen as opportunities for teachers and students to engage in the construction of models. In this process they pose and pursue relevant questions as a mean to identify and represent relevant information that guides that construction. In this study, high school teachers worked on a series of mathematical tasks in which they had the opportunity to construct and explore mathematical models. Those models provided them relevant information to think of and design their instructional routes. They were encouraged to use dynamic software during the process of constructing and refining the models.

Conceptual Framework

Kelly and Lesh (2000) have recognized that researchers, teachers, and students rely on models to represent, organize, examine, and explain situations. For instance, researchers construct models to analyze and interpret teachers and students' activities. Teachers use models to describe, examine, and predict students' mathematical behaviors, while students use models to describe, explain, justify, and refine their ways of thinking. Thus, a model is conceived of as a conceptual unit or entity to foster and document both the teachers' construction of instructional routes and the students' development of mathematical knowledge.

In this context, it becomes important to identify not only the basic ingredients or elements of a model; but also to characterize the process involved in the construction of models. Doerr and English (2003, p. 112) define models as "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system". That is, the models construction involves examining the situation or problem to be modeled in order to identify essential elements that are represented and scrutinized through operations and rules with the aim of identifying and exploring mathematical relations. Here, we are interested in documenting cognitive behaviors that the problem solver (teacher or student) exhibits during the interaction with the task. Thus, it is important to distinguish phases or cycles that explain relevant moments around the teachers or students' process of models construction. In particular, ways in which teachers or students refine or transform initial models of the situation or phenomenon into more robust or improved models to deal with the situation.

In order to identify the essential elements embedded in a task or phenomenon it is important to comprehend initially the situation or problem (Polya, 1945). Understanding phenomena or situations that involve real contexts demands not only the recognition of the key elements around the problem or dilemma; but also ways to represent them mathematically. This phase is crucial to construct the model that will be explored through mathematical resources and strategies. The exploration stage leads us to search for different approaches and media to examine the model and eventually to solve the problem. The next stage is to interpret and validate the solution in terms of the original statement or problem conditions. In this process, it is important to analyze and discuss whether the model used to solve the problem or situation represents a tool to approach a family of problems or situations.

Is the model of the situation appropriate? Is the solution reasonable and consistent with the problem statement? Can the model be improved? Can the model be extended? What are the mathematical resources, concepts and strategies that were relevant during the construction and exploration of the model? These types of questions are crucial to evaluate the strengths and limitations of the model and to extend the model scope (Niss, Blum, & Galbraith, 2007).

Figure 1 represents a modeling cycle that shows relations and operations that allow transferring features of the phenomenon into the model construction. How can the use of computational tools influence and help the problem solver construct and explore mathematical models? Zbiek, Heid, and Blume, (2007, p. 1170) suggest that in experimental mathematics, computational tools can be used for:

(a) Gaining insight and intuition, (b) discovering new patterns and relationships, (c) graphing to expose mathematical principles, (d) testing and especially falsifying conjectures, (e) exploring a possible result to see whether it merits formal proof, (f) suggesting approaches for formal proof, (g) replacing lengthy hand derivations with tool computations, and (h) confirming analytically derived results.

In particular, the use of dynamic software could play an important role in constructing dynamic models of situations or tasks. The models represent configurations made of simple mathematical objects (points, segments, lines, triangles, squares, etc.) in which, some elements of the models can be moved within the configuration in order to identify and explore mathematical relations. As a consequence, the same process of model construction and exploration incorporates new ways to represent, formulate, and explore mathematical relations.

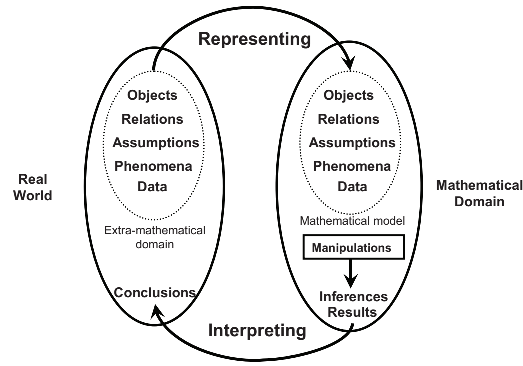

Figure 1. Modeling cycle.

For example, the situations or problems are now analyzed in terms of the facilities offered by the tool such as dragging particular components; find loci of points or lines, quantifying certain relations, etc. Indeed, the use of the tool offers the problem solvers the opportunity of exploring new routes to develop or reconstruct and explore basic mathematical results. In particular, the visual approach becomes relevant to identify the relations that later can be analyzed in terms of numeric and graphic approaches.

The research questions used to guide and structure the development of the study were: (a) what ways of reasoning or thinking about the problems emerge during the model construction with the use of the tools?, (b) to what extent does the use of computational tools offer teachers the possibility of constructing dynamic models of problems to identify and explore diverse mathematical relations?, and (c) what types of mathematical resources and strategies emerge during the teachers' construction of mathematical models associated with the phenomena?

Research Design, Methods and General Procedures

Six high school teachers participated in 3 hours weekly problem-solving sessions during one semester. The teachers were encouraged to use dynamic software (Cabri-Geometry) to construct dynamic models associated with the tasks or situations. Some problems were selected from textbooks or research articles and others came from the teachers' own construction of geometric configurations with the use of the tool. In general, the didactic approach during the sessions involved working in pairs and plenary presentations. Two researchers coordinated the development of the sessions and participated as members of a community that fostered an

inquisitive approach to the tasks. In this process, the teachers worked as a part of the community not only to solve the problems; but also to have opportunities to review mathematical contents that emerged while solving the tasks. The problem solving sessions were recorded and each team handed in a report that included the software files. The researchers took notes and discussed the advantages in using the tool during the diverse problem solving phases. In this report, we focus on analyzing the work shown by the community while dealing with a problem embedded in a real context. Thus, the unit of analysis is the work shown by the six participants as a group during the sessions. The task discussed throughout this report is representative of the type of problems that the community addressed during the development of the sessions.
In general, the participants had experience in using computational tools and they were encouraged to use them during their interaction with the tasks.

The task. Figure 2 shows a car going on a straight highway. Aside there is a palace and the driver wants to stop so that his friend (the passenger) can appreciate the facade of the palace. At what position of the highway should the driver stop the car, so that his friend can have the best view? (Adjusted from Vasíliev & Gutenmájer, 1980).

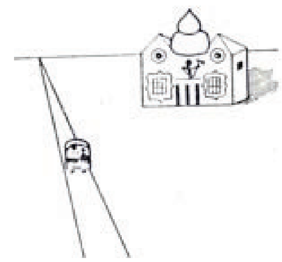

Figure 2. The palace.

Presentation of Results

We organize and structure the results in terms of identifying essential phases around the process of model construction that the participants exhibited during the interaction with the task. These phases include the initial comprehension of the statement of the task; the identification of basic elements to construct a model of the problem; the exploration of the model and the formulation of conjectures; ways to support mathematical relations and conjectures, interpretation of results and validation and extension of the model.

A. Understanding the task statement: An inquiry approach. This phase was important to comprehend the task and to identify and discuss a set of assumptions that led the community to identify the elements to be considered in the model construction. To this end, the community posed and discussed the following questions: How can we identify that for a distinct position of the car, the passenger has different views of the palace? What does it mean to have the best view? Is it sufficient to consider the position of the passenger on the highway as a reference instead of the car to determine the best position? It was observed that the figure provided in the statement of the task helped them to visualize and eventually represent the problem. They relied on the figure to assume that the observer could be identified as one point that is moved along a line (the highway). Here, the community also discussed if the provided information was sufficient to solve the task since there was no quantitative data involved in the statement.

Initially, the community identified two ways to characterize the best view of the facade: One that related the distance between the observer and the facade (less distance better view) and the other that focused on relating the best view to the angle that is formed between two points on the facade and the observer. The former interpretation was chosen by the community and became a source or a departure point to construct the model of the situation. At this moment, the task was thought of in terms of the basic elements (line, angles, segments, etc) as a way to construct a mathematical model.

Swars, S. L., Stinson, D. W., & Lemons-Smith, S. (Eds.). (2009). Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Atlanta, GA: Georgia State University.

B. Model construction. The initial analysis of the statement led the community to construct a dynamic model of the situation by representing elements of the task (highway, facade, and observer) through geometric objects (lines, segments, points, angles). In this context, Sophia proposed to represent the highway with a straight line and the passenger a point of that line, and a segment as the base of the palace's facade. Why can the facade of the palace be represented through a segment? Some of the participants argued that the best view means to compare angles that relate the wide of the palace (represented by a segment) and the point that represents the observer. In general, the participants agreed that Sophia's representation included the relevant information of the task. It is important to mention that in order to explain what happens to the angle for various position of the point (the observer) they recognized that it was necessary to identify and notate explicitly the main objects embedded in the problem (points, angles, line, segment). Here, some teachers initially used paper and pencil to sketch a problem representation but later, the use of the tool (dynamic software) became important to visualize the angle variation for different positions of the observer.

C. Model exploration and conjectures. At this stage, there appeared two ways to represent the statement: One in which some participants used paper and pencil and relied on trigonometric relations to construct and explore the model (Figure 3, left); and the second approach in which the use of the tool guided the model exploration. Thus, the participants who decided to use the software, started to observe the behavior of some attributes of triangle APB (area, perimeter, and angles) when point P was moved along line L. Here, by assigning measurements to those attributes, both the areas and perimeters of the family of triangles did not reach a maximum value. In particular, Sophia and Jacob noticed that when point P was situated on a position that was collinear with point A and B, then the angle APB measured zero degrees; but when point P was moved to the right of the collinear position the measurement of angle APB increased for some positions and then the angle value decreased.

Figure 3. Paper and pencil representation (left) and a dynamic model (right).

Thus, they focused on determining the position for point P on L in which the angle APB reaches its maximum value. There appeared two ways to identify the maximum value of the angle. One in which some of the participants directly visualize the numbers displayed while moving point P along the line L (Figure 4); and another in which the participants construct a graphic representation that involves the distance AP and the corresponding value of angle APB (Figure 5).

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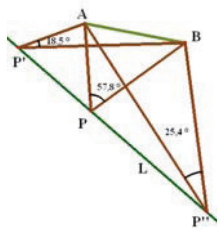

Figure 4. Identifying various positions for point P and the angles formed with end points of segment AB.

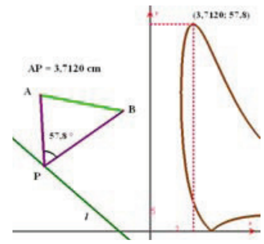

Figure 5. Graphic representation of the variation of the value of angle ABP when point P is moved along line L.

The participants were aware of the need of looking for an algebraic or geometric argument to justify the position of point P where the angle reaches its maximum value. In this process, Daniel and Emily decided to draw the circle that passes through points P, A, and B. Based on this construction, they realized that when point P is moved along line L, then the circle that passes through points P, A, and B seems to be tangent to line L at the position where angle APB reaches its maximum value (Figure 6).

Based on this information a conjecture emerged: *To identify the point where angle APB reaches its maximum value is sufficient to draw a tangent circle to line L that passes through points A and B. That is, the tangency point of the circle and line L is the place where the observer gets the best view of the palace*. How can we construct the circle that passes through A and B and is tangent to line L? Emily posed this question to the rest of the participants during the class discussion. Sophia and Jacob suggested that it was relevant to identify relevant properties of the tangent circle assuming its existence. That is, if the tangent circle exists, what properties should it have? Here, it was recognized that the circle must lie on the perpendicular bisector of segment AB and also that its centre must also be on the perpendicular to line L that passes through the tangent point. Thus, David drew a perpendicular line to L that passes through point P and the perpendicular bisector of segment AP. These lines get intersected at point D. What is the locus of point D when point P is moved along line L? With the use of the software the locus was determined (Figure 7). Thus, the intersection point (C) of the locus and the perpendicular bisector of AB was the centre of the tangent circle. Here, to draw the circle they drew the perpendicular from point C to line L, and the distance from point C to line L was the radius of the tangent circle (Figure 7). During the class, it was also argued that the locus of point D when point P is moved along line L is a parabola, since point D is on the perpendicular bisector of segment AP and it holds that d(P,D) is always the same as d(D,A) (definition of perpendicular bisector). Here, the focus of the parabola is point A and the directrix is the line L.

Sophia and Jacob constructed the tangent circle to L that passes through point A and B by drawing initially the perpendicular bisector of segment AB. Later, they situated a point C on that perpendicular bisector and drew a circle with center point C and radius the segment CA. They also drew a perpendicular to line L that passes through point C. This perpendicular and the circle get intersected at point D. What is the locus of point D when point C is moved along the perpendicular bisector of AB?

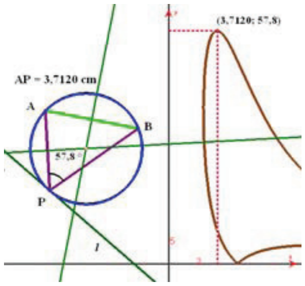

Figure 6. The circle that passes through point P, A, and B seems to be tangent to line L when angle APB gets the maximum value.

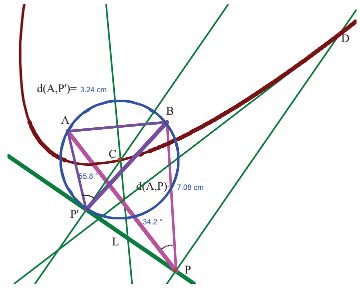

Figure 7. Drawing a tangent circle to line L that passes through points A and B.

Again, the use of the software showed that such locus was one branch of a hyperbola. The locus intersects line L at point P. The perpendicular line to L that passes through point P intersects the perpendicular bisector of segment AB at point C'. Thus, to draw the tangent circle to L that passes through points A and B it was sufficient to draw the circle with center point C' and radius segment C'P (Figure 8).

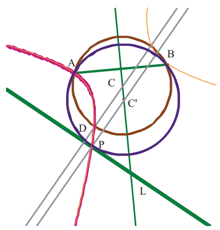

Figure 8. Using a hyperbola to construct a tangent circle to line L that passes through points A and B.

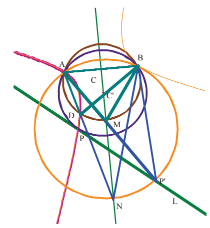

Figure 9. Providing an argument to show that angle APB reaches the maximum value.

D. Interpretation and model validation. During the class discussion, the participants recognized that the problem of finding the best view was reduced to construct a tangent circle to a line that passes through two given points; however, it was important to provide a mathematical argument to validate that the tangency point was the position where the angle gets its maximum value. To present the argument they relied on figure 9: Point M and N are the intersection points of the perpendicular bisector of segment AB and circles that pass through points ABD and ABP' respectively.

Thus, to compare the values of angle ADB and angle AP'B is the same as comparing angles AMB and ANB. This is because angle ADB is congruent with angle AMB and angle ANB is congruent with angle AP'B. It is also observed that d(A, N) becomes equal to AP when D coincides with point P (tangency point), otherwise d(A, N) is always larger than AM. Therefore, the tangency angle is the angle with maximum value. To evaluate the appropriateness and feasibility of the model the participants changed the original position of the essential elements (highway and facade) and they observed that the model also allowed them to identify the best view of the facade. Including the case in which the facade (segment AB) and the highway (line

L) are parallel, here the best view appears at the intersection of the perpendicular bisector of segment AB and line L.

The participants observed that the domain of the solutions lies on the interval between the intersection of the perpendicular from the extreme of the facade that is closest to the line (highway) and the intersection point of the perpendicular bisector of segment AB and the highway (Figure 10).

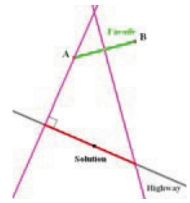

Figure 10. The model's domain.

Discussion and Remarks

The model approach used to guide the development of the problem solving sessions helped the participants to focus on key aspects associated with the development of mathematical thinking and practice. For example, firstly, the participants, working as a part of a community, realized that the process of initially comprehending the problem statements is crucial not only to identify essential aspects of the situation, but also to recognize a series of assumptions needed to construct a model of the task or problem. Secondly, they recognized that the model exploration phase represents a departure point for the problem solver to examine the model from distinct perspectives with the aim of identifying a set of relations or conjectures. Later, the conjectures that emerge, during the model exploration phase, need to be supported with mathematical arguments. Finally, the model used to solve the problem needs to be examined in order to evaluate and contrast its pertinence and possible extension to be used in isomorphic or related tasks.

In this context, there is evidence that the use of the tool helped the participants to initially construct a dynamic model of the task. Thus, moving a point (P) on a line (highway) led them to identify and relate the "best view" with the angle formed between the ends of a segment (palace) and that point. How can we measure the angle for distinct positions of point P? How can we identify the angle with a largest value? The teachers used the software to measure the angle for various positions of point P to observe that there was a position where the angle's value was the largest. This visual and empirical approach became important to think of other ways to represent the angle variation. The graphic solution involved a functional approach in which the use of a Cartesian system helped them relate the distance from one end of the segment (AB, the palace) and its corresponding angle. Thus, the graphic approach became relevant to visually identify the point where the angle reaches its maximum value. In addition, moving the point P on the line helped the participants to observe particular behaviors of other objects (circles, segments) within the representations. For example, the teachers observed that the circle that passes through the three points (A, B, and P) becomes tangent to the line when the angle APB reaches its maximum value. Thus, the solution of the task was reduced to draw a circle tangent to the line and the tangency point was the desired point. Again, analyzing relevant properties of the possible solution led them to construct the perpendicular bisector of segment AB and the perpendicular line to L that passes through point P. The locus of the intersection point of those lines when P is moved through line L generated a parabola. Here, the parabola was the key to find the solution of the task. The participants were surprised that a conic section was used to find the point where the observer gets the best view of the palace. They also recognized that the dynamic representation of the problem became important to identify two mathematical results: (a) given a line L and a segment AB that is not parallel to line L, then there is a point P' on the line where the circle that passes through points A, B and P' is tangent to line L. Here the angle AP'B is the angle with the maximum value, (b) given a line L, a point P on that line, and a segment AB that is not parallel to L, then the locus of the intersection point of the perpendicular bisector of segment AP (or BP) and the perpendicular line to L that passes through point P when point P is moved along line L is a parabola. The modeling phases, described in this report, provided useful information to identify a potential route for students to approach mathematical tasks with the use of the tool. Thus, the construction of the dynamic representation, the quantification of attributes (measures of segments, angles, etc.), the identification of loci and the graphic representations are key activities that can help teachers and students to identify and explore interesting mathematical relations. In addition, the use of the tools is also relevant to search for arguments to support those results.

In short, during the modeling processes the participants had an opportunity of identifying and discussing assumptions and essential components that were relevant to construct a dynamic model of the task. The exploration of the model, through an inquisitive approach, led them to formulate a set of conjectures and relations that were important during the solution process. To reach the solution, they relied on empirical, numeric, visual, and algebraic approaches to support and validate conjectures. In this context, the use of the tool seems to help the teachers to experience themselves diverse routes to reconstruct basic mathematical results. These routes are key ingredients for teachers to identify instructional strategies that can foster their students' development of mathematical thinking.

Acknowledgement

We acknowledge the support received by Conacyt (reference No. 47850), during the development of this research.

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